

Superposition of Two Perpendicular Simple Harmonic Vibrations

(1) Vibrations Having Equal Frequencies

Suppose that a particle moves under the simultaneous influence of two simple harmonic vibrations of equal frequency, one along the x axis, the other along the perpendicular y axis. What is its subsequent motion?

This displacements may be written

$$x = a_1 \sin(\omega t + \phi_1)$$

$$y = a_2 \sin(\omega t + \phi_2)$$

and the path followed by the particle is formed by eliminating the time t from these equations to leave an expression involving only x and y and the constants ϕ_1 and ϕ_2 .

Expanding the arguments of the sines we have

$$\frac{x}{a_1} = \sin \omega t \cos \phi_1 + \cos \omega t \sin \phi_1$$

and

$$\frac{y}{a_2} = \sin \omega t \cos \phi_2 + \cos \omega t \sin \phi_2$$

If we carry out the process

$$\left(\frac{x}{a_1} \sin \phi_2 - \frac{y}{a_2} \sin \phi_1 \right)^2 + \left(\frac{y}{a_2} \cos \phi_1 - \frac{x}{a_1} \cos \phi_2 \right)^2$$

this will yield

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2} \cos(\phi_2 - \phi_1) = \sin^2(\phi_2 - \phi_1) \quad (1.3)$$

which is the general equation for an ellipse.

In the most general case the axes of the ellipse are inclined to the x and y axes, but these become the principal axes when the phase difference

$$\phi_2 - \phi_1 = \frac{\pi}{2}$$

Equation (1.3) then takes the familiar form

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1$$

that is, an ellipse with semi-axes a_1 and a_2 .

If $a_1 = a_2 = a$ this becomes the circle

$$x^2 + y^2 = a^2$$

When

$$\phi_2 - \phi_1 = 0, 2\pi, 4\pi, \text{ etc.}$$

the equation simplifies to

$$y = \frac{a_2}{a_1}x$$

which is a straight line through the origin of slope a_2/a_1 .

Again for $\phi_2 - \phi_1 = \pi, 3\pi, 5\pi, \text{ etc.}$, we obtain

$$y = -\frac{a_2}{a_1}x$$

a straight line through the origin of equal but opposite slope.

The paths traced out by the particle for various values of $\delta = \phi_2 - \phi_1$ are shown in Figure 1.8 and are most easily demonstrated on a cathode ray oscilloscope.

When

$$\phi_2 - \phi_1 = 0, \pi, 2\pi, \text{ etc.}$$

and the ellipse degenerates into a straight line, the resulting vibration lies wholly in one plane and the oscillations are said to be *plane polarized*.

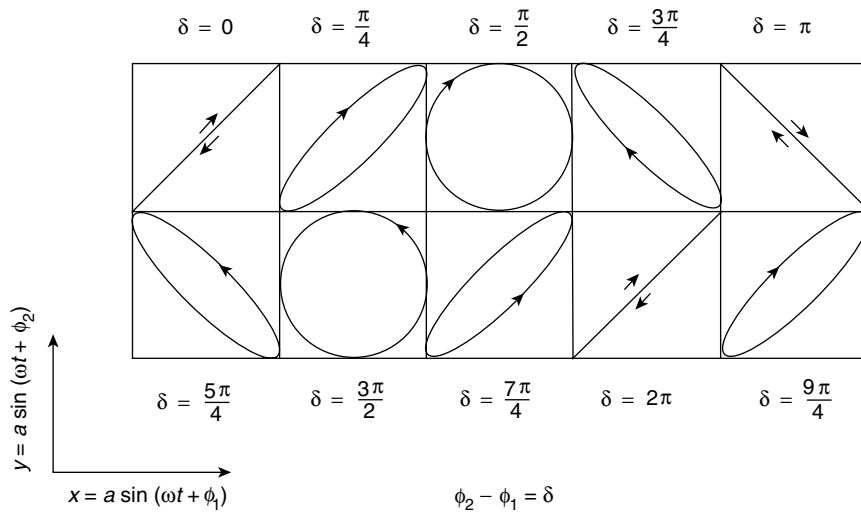


Figure 1.8 Paths traced by a system vibrating simultaneously in two perpendicular directions with simple harmonic motions of equal frequency. The phase angle δ is the angle by which the y motion leads the x motion

Convention defines the plane of polarization as that plane perpendicular to the plane containing the vibrations. Similarly the other values of

$$\phi_2 - \phi_1$$

yield *circular* or *elliptic* polarization where the tip of the vector resultant traces out the appropriate conic section.

(Problems 1.14, 1.15, 1.16)

*Polarization

Polarization is a fundamental topic in optics and arises from the superposition of two perpendicular simple harmonic optical vibrations. We shall see in Chapter 8 that when a light wave is plane polarized its electrical field oscillation lies within a single plane and traces a sinusoidal curve along the direction of wave motion. Substances such as quartz and calcite are capable of splitting light into two waves whose planes of polarization are perpendicular to each other. Except in a specified direction, known as the optic axis, these waves have different velocities. One wave, the ordinary or *O* wave, travels at the same velocity in all directions and its electric field vibrations are always perpendicular to the optic axis. The extraordinary or *E* wave has a velocity which is direction-dependent. Both ordinary and extraordinary light have their own refractive indices, and thus quartz and calcite are known as doubly refracting materials. When the ordinary light is faster, as in quartz, a crystal of the substance is defined as positive, but in calcite the extraordinary light is faster and its crystal is negative. The surfaces, spheres and ellipsoids, which are the loci of the values of the wave velocities in any direction are shown in Figure 1.9(a), and for a

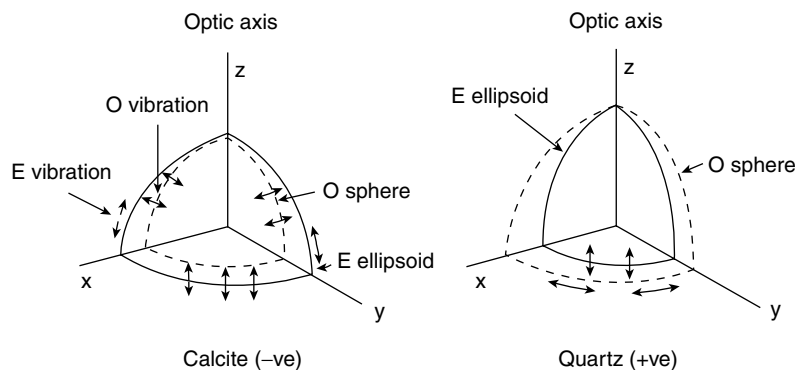


Figure 1.9a Ordinary (spherical) and extraordinary (ellipsoidal) wave surfaces in doubly refracting calcite and quartz. In calcite the *E* wave is faster than the *O* wave, except along the optic axis. In quartz the *O* wave is faster. The *O* vibrations are always perpendicular to the optic axis, and the *O* and *E* vibrations are always tangential to their wave surfaces

*This section may be omitted at a first reading.

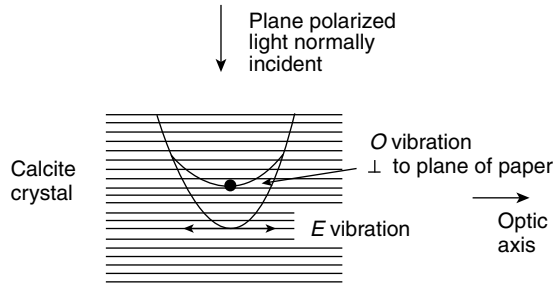


Figure 1.9b Plane polarized light normally incident on a calcite crystal face cut parallel to its optic axis. The advance of the E wave over the O wave is equivalent to a gain in phase

given direction the electric field vibrations of the separate waves are tangential to the surface of the sphere or ellipsoid as shown. Figure 1.9(b) shows plane polarized light normally incident on a calcite crystal cut parallel to its optic axis. Within the crystal the faster E wave has vibrations parallel to the optic axis, while the O wave vibrations are perpendicular to the plane of the paper. The velocity difference results in a phase gain of the E vibration over the O vibration which increases with the thickness of the crystal. Figure 1.9(c) shows plane polarized light normally incident on the crystal of Figure 1.9(b) with its vibration at an angle of 45° of the optic axis. The crystal splits the vibration into

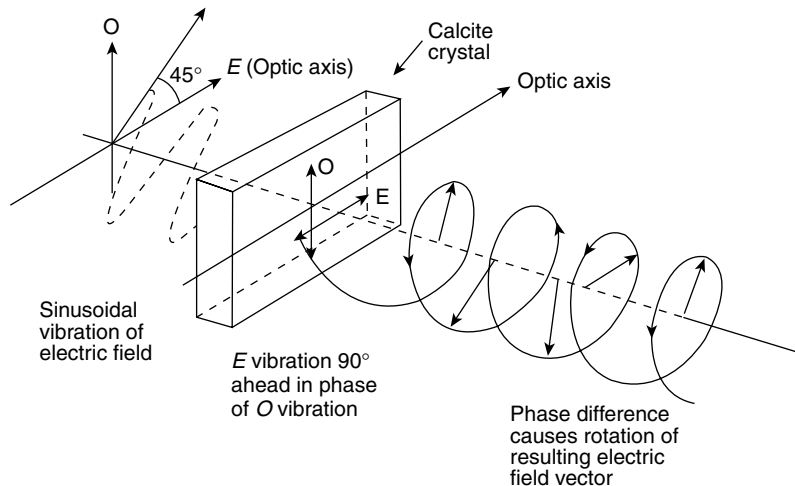


Figure 1.9c The crystal of Fig. 1.9c is thick enough to produce a phase gain of $\pi/2$ rad in the E wave over the O wave. Wave recombination on leaving the crystal produces circularly polarized light

equal E and O components, and for a given thickness the E wave emerges with a phase gain of 90° over the O component. Recombination of the two vibrations produces circularly polarized light, of which the electric field vector now traces a helix in the anticlockwise direction as shown.

(2) *Vibrations Having Different Frequencies (Lissajous Figures)*

When the frequencies of the two perpendicular simple harmonic vibrations are not equal the resulting motion becomes more complicated. The patterns which are traced are called Lissajous figures and examples of these are shown in Figure 1.10 where the axial frequencies bear the simple ratios shown and

$$\begin{aligned} \delta &= \phi_2 - \phi_1 = 0 \text{ (on the left)} \\ &= \frac{\pi}{2} \text{ (on the right)} \end{aligned}$$

If the amplitudes of the vibrations are respectively a and b the resulting Lissajous figure will always be contained within the rectangle of sides $2a$ and $2b$. The sides of the rectangle will be tangential to the curve at a number of points and the ratio of the numbers of these tangential points along the x axis to those along the y axis is the inverse of the ratio of the corresponding frequencies (as indicated in Figure 1.10).

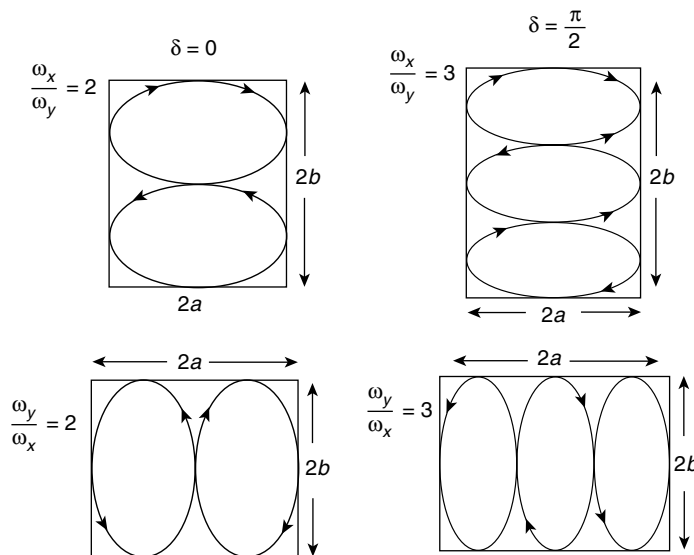


Figure 1.10 Simple Lissajous figures produced by perpendicular simple harmonic motions of different angular frequencies

Superposition of a Large Number n of Simple Harmonic Vibrations of Equal Amplitude a and Equal Successive Phase Difference δ

Figure 1.11 shows the addition of n vectors of equal length a , each representing a simple harmonic vibration with a constant phase difference δ from its neighbour. Two general physical situations are characterized by such a superposition. The first is met in Chapter 5 as a wave group problem where the phase difference δ arises from a small *frequency difference*, $\delta\omega$, between consecutive components. The second appears in Chapter 12 where the intensity of optical interference and diffraction patterns are considered. There, the superposed harmonic vibrations will have the same frequency but each component will have a constant phase difference from its neighbour because of the extra *distance* it has travelled.

The figure displays the mathematical expression

$$R \cos(\omega t + \alpha) = a \cos \omega t + a \cos(\omega t + \delta) + a \cos(\omega t + 2\delta) \\ + \cdots + a \cos(\omega t + [n - 1]\delta)$$

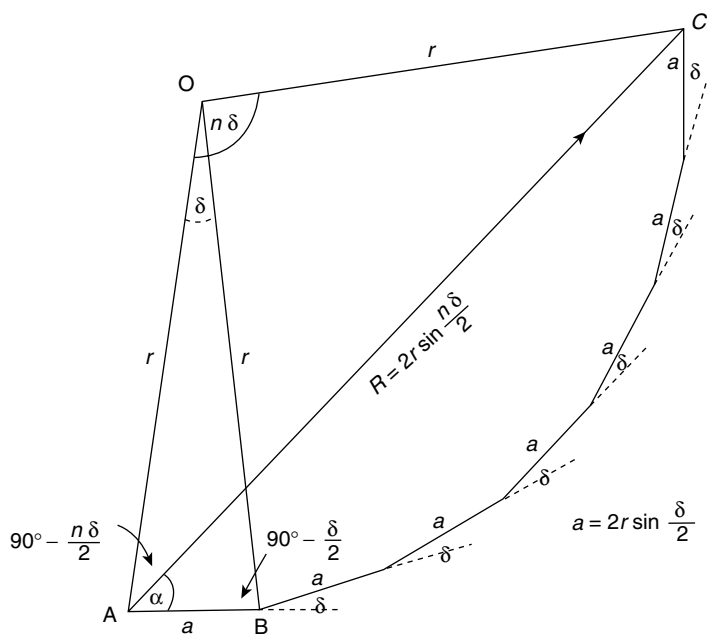


Figure 1.11 Vector superposition of a large number n of simple harmonic vibrations of equal amplitude a and equal successive phase difference δ . The amplitude of the resultant

$$R = 2r \sin \frac{n\delta}{2} = a \frac{\sin n\delta/2}{\sin \delta/2}$$

and its phase with respect to the first contribution is given by

$$\alpha = (n - 1)\delta/2$$

where R is the magnitude of the resultant and α is its phase difference with respect to the first component $a \cos \omega t$.

Geometrically we see that each length

$$a = 2r \sin \frac{\delta}{2}$$

where r is the radius of the circle enclosing the (incomplete) polygon.

From the isosceles triangle OAC the magnitude of the resultant

$$R = 2r \sin \frac{n\delta}{2} = a \frac{\sin n\delta/2}{\sin \delta/2}$$

and its phase angle is seen to be

$$\alpha = \widehat{OAB} - \widehat{OAC}$$

In the isosceles triangle OAC

$$\widehat{OAC} = 90^\circ - \frac{n\delta}{2}$$

and in the isosceles triangle OAB

$$\widehat{OAB} = 90^\circ - \frac{\delta}{2}$$

so

$$\alpha = \left(90^\circ - \frac{\delta}{2}\right) - \left(90^\circ - \frac{n\delta}{2}\right) = (n-1) \frac{\delta}{2}$$

that is, half the phase difference between the first and the last contributions. Hence the resultant

$$R \cos (\omega t + \alpha) = a \frac{\sin n\delta/2}{\sin \delta/2} \cos \left[\omega t + (n-1) \frac{\delta}{2} \right]$$

We shall obtain the same result later in this chapter as an example on the use of exponential notation.

For the moment let us examine the behaviour of the magnitude of the resultant

$$R = a \frac{\sin n\delta/2}{\sin \delta/2}$$

which is not constant but depends on the value of δ . When n is very large δ is very small and the polygon becomes an arc of the circle centre O , of length $na = A$, with R as the chord. Then

$$\alpha = (n-1) \frac{\delta}{2} \approx \frac{n\delta}{2}$$

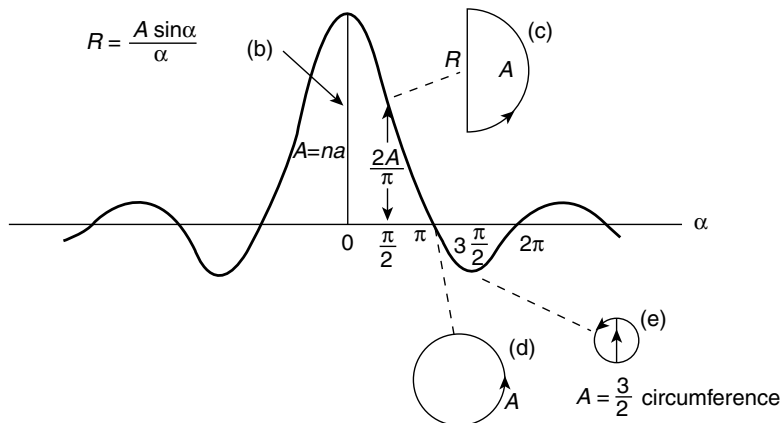


Figure 1.12 (a) Graph of $A \sin \alpha/\alpha$ versus α , showing the magnitude of the resultants for (b) $\alpha = 0$; (c) $\alpha = \pi/2$; (d) $\alpha = \pi$ and (e) $\alpha = 3\pi/2$

and

$$\sin \frac{\delta}{2} \rightarrow \frac{\delta}{2} \approx \frac{\alpha}{n}$$

Hence, in this limit,

$$R = a \frac{\sin n\delta/2}{\sin \delta/2} = a \frac{\sin \alpha}{\alpha/n} = na \frac{\sin \alpha}{\alpha} = \frac{A \sin \alpha}{\alpha}$$

The behaviour of $A \sin \alpha/\alpha$ versus α is shown in Figure 1.12. The pattern is symmetric about the value $\alpha = 0$ and is zero whenever $\sin \alpha = 0$ except at $\alpha \rightarrow 0$ that is, when $\sin \alpha/\alpha \rightarrow 1$. When $\alpha = 0$, $\delta = 0$ and the resultant of the n vectors is the straight line of length A , Figure 1.12(b). As δ increases A becomes the arc of a circle until at $\alpha = \pi/2$ the first and last contributions are out of phase ($2\alpha = \pi$) and the arc A has become a semicircle of which the diameter is the resultant R Figure 1.12(c). A further increase in δ increases α and curls the constant length A into the circumference of a circle ($\alpha = \pi$) with a zero resultant, Figure 1.12(d). At $\alpha = 3\pi/2$, Figure 1.12(e) the length A is now $3/2$ times the circumference of a circle whose diameter is the amplitude of the first minimum.

*Superposition of n Equal SHM Vectors of Length a with Random Phase

When the phase difference between the successive vectors of the last section may take random values ϕ between zero and 2π (measured from the x axis) the vector superposition and resultant R may be represented by Figure 1.13.

*This section may be omitted at a first reading.

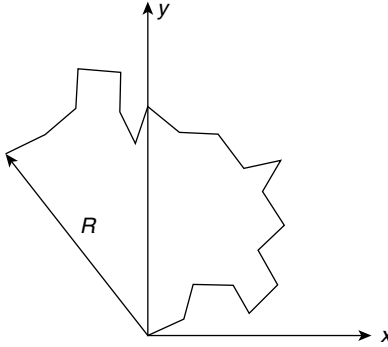


Figure 1.13 The resultant $R = \sqrt{na}$ of n vectors, each of length a , having random phase. This result is important in optical incoherence and in energy loss from waves from random dissipation processes

The components of R on the x and y axes are given by

$$R_x = a \cos \phi_1 + a \cos \phi_2 + a \cos \phi_3 \dots a \cos \phi_n$$

$$= a \sum_{i=1}^n \cos \phi_i$$

and

$$R_y = a \sum_{i=1}^n \sin \phi_i$$

where

$$R^2 = R_x^2 + R_y^2$$

Now

$$R_x^2 = a^2 \left(\sum_{i=1}^n \cos \phi_i \right)^2 = a^2 \left[\sum_{i=1}^n \cos^2 \phi_i + \sum_{\substack{i=1 \\ i \neq j}}^n \cos \phi_i \sum_{j=1}^n \cos \phi_j \right]$$

In the typical term $2 \cos \phi_i \cos \phi_j$ of the double summation, $\cos \phi_i$ and $\cos \phi_j$ have random values between ± 1 and the averaged sum of sets of these products is effectively zero.

The summation

$$\sum_{i=1}^n \cos^2 \phi_i = n \overline{\cos^2 \phi}$$

that is, the number of terms n times the average value $\overline{\cos^2 \phi}$ which is the integrated value of $\cos^2 \phi$ over the interval zero to 2π divided by the total interval 2π , or

$$\overline{\cos^2 \phi} = \frac{1}{2\pi} \int_0^{2\pi} \cos^2 \phi \, d\phi = \frac{1}{2} = \overline{\sin^2 \phi}$$

So

$$R_x^2 = a^2 \sum_{i=1}^n \cos^2 \phi_i = na^2 \overline{\cos^2 \phi_i} = \frac{na^2}{2}$$

and

$$R_y^2 = a^2 \sum_{i=1}^n \sin^2 \phi_i = na^2 \overline{\sin^2 \phi_i} = \frac{na^2}{2}$$

giving

$$R^2 = R_x^2 + R_y^2 = na^2$$

or

$$R = \sqrt{na}$$

Thus, the amplitude R of a system subjected to n equal simple harmonic motions of amplitude a with random phases is only \sqrt{na} whereas, if the motions were all in phase R would equal na .

Such a result illustrates a very important principle of random behaviour.

(Problem 1.17)

Applications

Incoherent Sources in Optics The result above is directly applicable to the problem of coherence in optics. Light sources which are in phase are said to be coherent and this condition is essential for producing optical interference effects experimentally. If the amplitude of a light source is given by the quantity a its intensity is proportional to a^2 , n coherent sources have a resulting amplitude na and a total intensity n^2a^2 . Incoherent sources have random phases, n such sources each of amplitude a have a resulting amplitude \sqrt{na} and a total intensity of na^2 .

Random Processes and Energy Absorption From our present point of view the importance of random behaviour is the contribution it makes to energy loss or absorption from waves moving through a medium. We shall meet this in all the waves we discuss.

Random processes, for example collisions between particles, in Brownian motion, are of great significance in physics. Diffusion, viscosity or frictional resistance and thermal conductivity are all the result of random collision processes. These energy dissipating phenomena represent the transport of mass, momentum and energy, and change only in the direction of increasing disorder. They are known as ‘thermodynamically irreversible’ processes and are associated with the increase of entropy. Heat, for example, can flow only from a body at a higher temperature to one at a lower temperature. Using the earlier analysis where the length a is no longer a simple harmonic amplitude but is now the average distance a particle travels between random collisions (its mean free path), we see that after n such collisions (with, on average, equal time intervals between collisions) the particle will, on average, have travelled only a distance \sqrt{na} from its position at time $t = 0$, so that the distance travelled varies only with the square root of the time elapsed instead of being directly proportional to it. This is a feature of all random processes.

Not all the particles of the system will have travelled a distance \sqrt{na} but this distance is the most probable and represents a statistical average.

Random behaviour is described by the diffusion equation (see the last section of Chapter 7) and a constant coefficient called the diffusivity of the process will always arise. The dimensions of a diffusivity are always length²/time and must be interpreted in terms of a characteristic distance of the process which varies only with the square root of time.

Some Useful Mathematics

The Exponential Series

By a ‘natural process’ of growth or decay we mean a process in which a quantity changes by a constant fraction of itself in a given interval of space or time. A 5% per annum compound interest represents a natural growth law; attenuation processes in physics usually describe natural decay.

The law is expressed differentially as

$$\frac{dN}{N} = \pm \alpha dx \quad \text{or} \quad \frac{dN}{N} = \pm \alpha dt$$

where N is the changing quantity, α is a constant and the positive and negative signs represent growth and decay respectively. The derivatives dN/dx or dN/dt are therefore proportional to the value of N at which the derivative is measured.

Integration yields $N = N_0 e^{\pm \alpha x}$ or $N = N_0 e^{\pm \alpha t}$ where N_0 is the value at x or $t = 0$ and e is the exponential or the base of natural logarithms. The exponential series is defined as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

and is shown graphically for positive and negative x in Figure 1.14. It is important to note that whatever the form of the index of the logarithmic base e , it is the power to which the

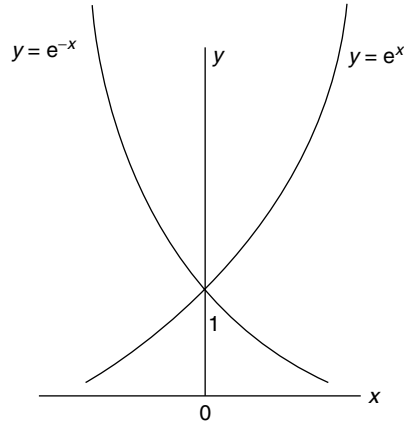


Figure 1.14 The behaviour of the exponential series $y = e^x$ and $y = e^{-x}$

base is raised, and is therefore always non-dimensional. Thus $e^{\alpha x}$ is non-dimensional and α must have the dimensions of x^{-1} . Writing

$$e^{\alpha x} = 1 + \alpha x + \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^3}{3!} + \dots$$

it follows immediately that

$$\begin{aligned} \frac{d}{dx}(e^{\alpha x}) &= \alpha + \frac{2\alpha^2}{2!}x + \frac{3\alpha^3}{3!}x^2 + \dots \\ &= \alpha \left[1 + \alpha x + \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^3}{3!} + \dots \right] \\ &= \alpha e^{\alpha x} \end{aligned}$$

Similarly

$$\frac{d^2}{dx^2}(e^{\alpha x}) = \alpha^2 e^{\alpha x}$$

In Chapter 2 we shall use $d(e^{\alpha t})/dt = \alpha e^{\alpha t}$ and $d^2(e^{\alpha t})/dt^2 = \alpha^2 e^{\alpha t}$ on a number of occasions.

By taking logarithms it is easily shown that $e^x e^y = e^{x+y}$ since $\log_e(e^x e^y) = \log_e e^x + \log_e e^y = x + y$.

The Notation $i = \sqrt{-1}$

The combination of the exponential series with the complex number notation $i = \sqrt{-1}$ is particularly convenient in physics. Here we shall show the mathematical convenience in expressing sine or cosine (oscillatory) behaviour in the form $e^{ix} = \cos x + i \sin x$.

In Chapter 3 we shall see the additional merit of i in its role of vector operator. The series representation of $\sin x$ is written

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

and that of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$$

Since

$$i = \sqrt{-1}, i^2 = -1, i^3 = -i$$

etc. we have

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \dots \\ &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) \\ &= \cos x + i \sin x \end{aligned}$$

We also see that

$$\frac{d}{dx}(e^{ix}) = i e^{ix} = i \cos x - \sin x$$

Often we shall represent a sine or cosine oscillation by the form e^{ix} and recover the original form by taking that part of the solution preceded by i in the case of the sine, and the real part of the solution in the case of the cosine.

Examples

(1) In simple harmonic motion ($\ddot{x} + \omega^2 x = 0$) let us try the solution $x = a e^{i\omega t} e^{i\phi}$, where a is a constant length, and ϕ (and therefore $e^{i\phi}$) is a constant.

$$\begin{aligned} \frac{dx}{dt} &= \dot{x} = i\omega a e^{i\omega t} e^{i\phi} = i\omega x \\ \frac{d^2x}{dt^2} &= \ddot{x} = i^2 \omega^2 a e^{i\omega t} e^{i\phi} = -\omega^2 x \end{aligned}$$

Therefore

$$\begin{aligned} x &= a e^{i\omega t} e^{i\phi} = a e^{i(\omega t + \phi)} \\ &= a \cos(\omega t + \phi) + i a \sin(\omega t + \phi) \end{aligned}$$

is a complete solution of $\ddot{x} + \omega^2 x = 0$.

On p. 6 we used the sine form of the solution; the cosine form is equally valid and merely involves an advance of $\pi/2$ in the phase ϕ .

(2)

$$\begin{aligned} e^{ix} + e^{-ix} &= 2 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) = 2 \cos x \\ e^{ix} - e^{-ix} &= 2i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) = 2i \sin x \end{aligned}$$

(3) On p. 21 we used a geometrical method to show that the resultant of the superposed harmonic vibrations

$$\begin{aligned} &a \cos \omega t + a \cos (\omega t + \delta) + a \cos (\omega t + 2\delta) + \dots + a \cos (\omega t + [n-1]\delta) \\ &= a \frac{\sin n\delta/2}{\sin \delta/2} \cos \left\{ \omega t + \left(\frac{n-1}{2} \right) \delta \right\} \end{aligned}$$

We can derive the same result using the complex exponential notation and *taking the real part* of the series expressed as the geometrical progression

$$\begin{aligned} &a e^{i\omega t} + a e^{i(\omega t + \delta)} + a e^{i(\omega t + 2\delta)} + \dots + a e^{i[\omega t + (n-1)\delta]} \\ &= a e^{i\omega t} (1 + z + z^2 + \dots + z^{(n-1)}) \end{aligned}$$

where $z = e^{i\delta}$.

Writing

$$S(z) = 1 + z + z^2 + \dots + z^{n-1}$$

and

$$z[S(z)] = z + z^2 + \dots + z^n$$

we have

$$S(z) = \frac{1 - z^n}{1 - z} = \frac{1 - e^{in\delta}}{1 - e^{i\delta}}$$

So

$$\begin{aligned} a e^{i\omega t} S(z) &= a e^{i\omega t} \frac{1 - e^{in\delta}}{1 - e^{i\delta}} \\ &= a e^{i\omega t} \frac{e^{in\delta/2} (e^{-in\delta/2} - e^{in\delta/2})}{e^{i\delta/2} (e^{-i\delta/2} - e^{i\delta/2})} \\ &= a e^{i[\omega t + (\frac{n-1}{2})\delta]} \frac{\sin n\delta/2}{\sin \delta/2} \end{aligned}$$

with the real part

$$= a \cos \left[\omega t + \left(\frac{n-1}{2} \right) \delta \right] \frac{\sin n\delta/2}{\sin \delta/2}$$

which recovers the original cosine term from the complex exponential notation.

(Problem 1.18)

(4) Suppose we represent a harmonic oscillation by the complex exponential form

$$z = a e^{i\omega t}$$

where a is the amplitude. Replacing i by $-i$ defines the *complex conjugate*

$$z^* = a e^{-i\omega t}$$

The use of this conjugate is discussed more fully in Chapter 3 but here we can note that the product of a complex quantity and its conjugate is always equal to the square of the amplitude for

$$\begin{aligned} z z^* &= a^2 e^{i\omega t} e^{-i\omega t} = a^2 e^{(i-i)\omega t} = a^2 e^0 \\ &= a^2 \end{aligned}$$

(Problem 1.19)

Problem 1.1

The equation of motion

$$m\ddot{x} = -sx \quad \text{with} \quad \omega^2 = \frac{s}{m}$$

applies directly to the system in Figure 1.1(c).

If the pendulum bob of Figure 1.1(a) is displaced a small distance x show that the stiffness (restoring force per unit distance) is mg/l and that $\omega^2 = g/l$ where g is the acceleration due to gravity. Now use the small angular displacement θ instead of x and show that ω is the same.

In Figure 1.1(b) the angular oscillations are rotational so the mass is replaced by the moment of inertia I of the disc and the stiffness by the restoring couple of the wire which is $C \text{ rad}^{-1}$ of angular displacement. Show that $\omega^2 = C/I$.

In Figure 1.1(d) show that the stiffness is $2T/l$ and that $\omega^2 = 2T/lm$.

In Figure 1.1(e) show that the stiffness of the system is $2\rho Ag$, where A is the area of cross section and that $\omega^2 = 2g/l$ where g is the acceleration due to gravity.

In Figure 1.1(f) only the gas in the flask neck oscillates, behaving as a piston of mass ρAl . If the pressure changes are calculated from the equation of state use the adiabatic relation $pV^\gamma = \text{constant}$ and take logarithms to show that the pressure change in the flask is

$$dp = -\gamma p \frac{dV}{V} = -\gamma p \frac{Ax}{V},$$

where x is the gas displacement in the neck. Hence show that $\omega^2 = \gamma p A / l \rho V$. Note that γp is the stiffness of a gas (see Chapter 6).

In Figure 1.1(g), if the cross-sectional area of the neck is A and the hydrometer is a distance x above its normal floating level, the restoring force depends on the volume of liquid displaced (Archimedes' principle). Show that $\omega^2 = g \rho A / m$.

Check the dimensions of ω^2 for each case.

Problem 1.2

Show by the choice of appropriate values for A and B in equation (1.2) that equally valid solutions for x are

$$x = a \cos(\omega t + \phi)$$

$$x = a \sin(\omega t - \phi)$$

$$x = a \cos(\omega t - \phi)$$

and check that these solutions satisfy the equation

$$\ddot{x} + \omega^2 x = 0$$

Problem 1.3

The pendulum in Figure 1.1(a) swings with a displacement amplitude a . If its starting point from rest is

$$(a) \quad x = a$$

$$(b) \quad x = -a$$

find the different values of the phase constant ϕ for the solutions

$$x = a \sin(\omega t + \phi)$$

$$x = a \cos(\omega t + \phi)$$

$$x = a \sin(\omega t - \phi)$$

$$x = a \cos(\omega t - \phi)$$

For each of the different values of ϕ , find the values of ωt at which the pendulum swings through the positions

$$x = +a/\sqrt{2}$$

$$x = a/2$$

and

$$x = 0$$

for the first time after release from

$$x = \pm a$$

Problem 1.4

When the electron in a hydrogen atom bound to the nucleus moves a small distance from its equilibrium position, a restoring force per unit distance is given by

$$s = e^2/4\pi\epsilon_0r^2$$

where $r = 0.05$ nm may be taken as the radius of the atom. Show that the electron can oscillate with a simple harmonic motion with

$$\omega_0 \approx 4.5 \times 10^{-16} \text{ rad s}^{-1}$$

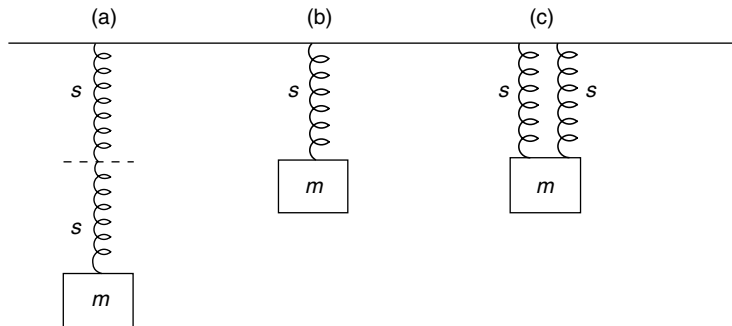
If the electron is forced to vibrate at this frequency, in which region of the electromagnetic spectrum would its radiation be found?

$$e = 1.6 \times 10^{-19} \text{ C, electron mass } m_e = 9.1 \times 10^{-31} \text{ kg}$$

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ N}^{-1} \text{ m}^{-2} \text{ C}^2$$

Problem 1.5

Show that the values of ω^2 for the three simple harmonic oscillations (a), (b), (c) in the diagram are in the ratio 1 : 2 : 4.



Problem 1.6

The displacement of a simple harmonic oscillator is given by

$$x = a \sin(\omega t + \phi)$$

If the oscillation started at time $t = 0$ from a position x_0 with a velocity $\dot{x} = v_0$ show that

$$\tan \phi = \omega x_0/v_0$$

and

$$a = (x_0^2 + v_0^2/\omega^2)^{1/2}$$

Problem 1.7

A particle oscillates with simple harmonic motion along the x axis with a displacement amplitude a and spends a time dt in moving from x to $x + dx$. Show that the probability of finding it between x and $x + dx$ is given by

$$\frac{dx}{\pi(a^2 - x^2)^{1/2}}$$

(in wave mechanics such a probability is not zero for $x > a$).

Problem. 1.8

Many identical simple harmonic oscillators are equally spaced along the x axis of a medium and a photograph shows that the locus of their displacements in the y direction is a sine curve. If the distance λ separates oscillators which differ in phase by 2π radians, what is the phase difference between two oscillators a distance x apart?

Problem 1.9

A mass stands on a platform which vibrates simple harmonically in a vertical direction at a frequency of 5 Hz. Show that the mass loses contact with the platform when the displacement exceeds 10^{-2} m.

Problem 1.10

A mass M is suspended at the end of a spring of length l and stiffness s . If the mass of the spring is m and the velocity of an element dy of its length is proportional to its distance y from the fixed end of the spring, show that the kinetic energy of this element is

$$\frac{1}{2} \left(\frac{m}{l} dy \right) \left(\frac{y}{l} v \right)^2$$

where v is the velocity of the suspended mass M . Hence, by integrating over the length of the spring, show that its total kinetic energy is $\frac{1}{6}mv^2$ and, from the total energy of the oscillating system, show that the frequency of oscillation is given by

$$\omega^2 = \frac{s}{M + m/3}$$

Problem 1.11

The general form for the energy of a simple harmonic oscillator is

$$E = \frac{1}{2} \text{mass (velocity)}^2 + \frac{1}{2} \text{stiffness (displacement)}^2$$

Set up the energy equations for the oscillators in Figure 1.1(a), (b), (c), (d), (e), (f) and (g), and use the expression

$$\frac{dE}{dt} = 0$$

to derive the equation of motion in each case.

Problem 1.12

The displacement of a simple harmonic oscillator is given by $x = a \sin \omega t$. If the values of the displacement x and the velocity \dot{x} are plotted on perpendicular axes, eliminate t to show that the locus of the points (x, \dot{x}) is an ellipse. Show that this ellipse represents a path of constant energy.

Problem 1.13

In Chapter 12 the intensity of the pattern when light from two slits interferes (Young's experiment) will be seen to depend on the superposition of two simple harmonic oscillations of equal amplitude a and phase difference δ . Show that the intensity

$$I = R^2 \propto 4a^2 \cos^2 \delta/2$$

Between what values does the intensity vary?

Problem 1.14

Carry out the process indicated in the text to derive equation (1.3) on p. 15.

Problem 1.15

The co-ordinates of the displacement of a particle of mass m are given by

$$x = a \sin \omega t$$

$$y = b \cos \omega t$$

Eliminate t to show that the particle follows an elliptical path and show by adding its kinetic and potential energy at any position x, y that the ellipse is a path of constant energy equal to the sum of the separate energies of the simple harmonic vibrations.

Prove that the quantity $m(x\dot{y} - y\dot{x})$ is also constant. What does this quantity represent?

Problem 1.16

Two simple harmonic motions of the same frequency vibrate in directions perpendicular to each other along the x and y axes. A phase difference

$$\delta = \phi_2 - \phi_1$$

exists between them such that the principal axes of the resulting elliptical trace are inclined at an angle to the x and y axes. Show that the measurement of two separate values of x (or y) is sufficient to determine the phase difference.

(Hint: use equation (1.3) and measure $y(\max)$, and y for $(x = 0)$.)

Problem 1.17

Take a random group of $n > 7$ values of ϕ in the range $0 \leq \phi \leq \pi$ and form the product

$$\sum_{\substack{i=1 \\ i \neq j}}^n \cos \phi_i \sum_{j=1}^n \cos \phi_j$$

Show that the average value obtained for several such groups is negligible with respect to $n/2$.

Problem 1.18

Use the method of example (3) (p. 28) to show that

$$\begin{aligned} a \sin \omega t + a \sin (\omega t + \delta) + a \sin (\omega t + 2\delta) + \cdots + a \sin [\omega t + (n-1)\delta] \\ = a \sin \left[\omega t + \frac{(n-1)\delta}{2} \right] \frac{\sin n\delta/2}{\sin \delta/2} \end{aligned}$$

Problem 1.19

If we represent the sum of the series

$$a \cos \omega t + a \cos (\omega t + \delta) + a \cos (\omega t + 2\delta) + \cdots + a \cos [\omega t + (n-1)\delta]$$

by the complex exponential form

$$z = a e^{i\omega t} (1 + e^{i\delta} + e^{i2\delta} + \cdots + e^{i(n-1)\delta})$$

show that

$$zz^* = a^2 \frac{\sin^2 n\delta/2}{\sin^2 \delta/2}$$

Summary of Important Results

Simple Harmonic Oscillator (mass m , stiffness s , amplitude a)

Equation of motion $\ddot{x} + \omega^2 x = 0$ where $\omega^2 = s/m$

Displacement $x = a \sin (\omega t + \phi)$

Energy $= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} s x^2 = \frac{1}{2} m \omega^2 a^2 = \frac{1}{2} s a^2 = \text{constant}$

Superposition (Amplitude and Phase) of two SHMs

One-dimensional

Equal ω , different amplitudes, phase difference δ , resultant R where $R^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos \delta$

Different ω , equal amplitude,

$$\begin{aligned} x &= x_1 + x_2 = a(\sin \omega_1 t + \sin \omega_2 t) \\ &= 2a \sin \frac{(\omega_1 + \omega_2)t}{2} \cos \frac{(\omega_2 - \omega_1)t}{2} \end{aligned}$$

Two-dimensional: perpendicular axes

Equal ω , different amplitude—giving general conic section

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2} \cos(\phi_2 - \phi_1) = \sin^2(\phi_2 - \phi_1)$$

(basis of optical polarization)

Superposition of n SHM Vectors (equal amplitude a , constant successive phase difference δ)

The resultant is $R \cos (\omega t + \alpha)$, where

$$R = a \frac{\sin n\delta/2}{\sin \delta/2}$$

and

$$\alpha = (n - 1)\delta/2$$

Important in optical diffraction and wave groups of many components.

2

Damped Simple Harmonic Motion

Initially we discussed the case of ideal simple harmonic motion where the total energy remained constant and the displacement followed a sine curve, apparently for an infinite time. In practice some energy is always dissipated by a resistive or viscous process; for example, the amplitude of a freely swinging pendulum will always decay with time as energy is lost. The presence of resistance to motion means that another force is active, which is taken as being proportional to the velocity. The frictional force acts in the direction opposite to that of the velocity (see Figure 2.1) and so Newton's Second law becomes

$$m\ddot{x} = -sx - r\dot{x}$$

where r is the constant of proportionality and has the dimensions of force per unit of velocity. The presence of such a term will always result in energy loss.

The problem now is to find the behaviour of the displacement x from the equation

$$m\ddot{x} + r\dot{x} + sx = 0 \quad (2.1)$$

where the coefficients m , r and s are constant.

When these coefficients are constant a solution of the form $x = Ce^{\alpha t}$ can always be found. Obviously, since an exponential term is always nondimensional, C has the dimensions of x (a length, say) and α has the dimensions of inverse time, T^{-1} . We shall see that there are three possible forms of this solution, each describing a different behaviour of the displacement x with time. In two of these solutions C appears explicitly as a constant length, but in the third case it takes the form

$$C = A + Bt^*$$

* The number of constants allowed in the general solution of a differential equation is always equal to the order (that is, the highest differential coefficient) of the equation. The two values A and B are allowed because equation (2.1) is second order. The values of the constants are adjusted to satisfy the initial conditions.

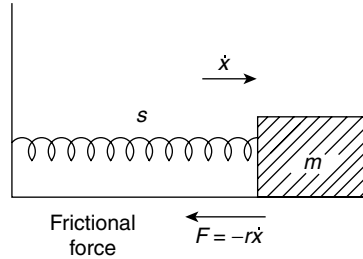


Figure 2.1 Simple harmonic motion system with a damping or frictional force $r\dot{x}$ acting against the direction of motion. The equation of motion is $m\ddot{x} + r\dot{x} + sx = 0$

where A is a length, B is a velocity and t is time, giving C the overall dimensions of a length, as we expect. From our point of view this case is not the most important.

Taking C as a constant length gives $\dot{x} = \alpha C e^{\alpha t}$ and $\ddot{x} = \alpha^2 C e^{\alpha t}$, so that equation (2.1) may be rewritten

$$C e^{\alpha t} (m\alpha^2 + r\alpha + s) = 0$$

so that either

$$x = C e^{\alpha t} = 0 \quad (\text{which is trivial})$$

or

$$m\alpha^2 + r\alpha + s = 0$$

Solving the quadratic equation in α gives

$$\alpha = \frac{-r}{2m} \pm \sqrt{\frac{r^2}{4m^2} - \frac{s}{m}}$$

Note that $r/2m$ and $(s/m)^{1/2}$, and therefore, α , all have the dimensions of inverse time, T^{-1} , which we expect from the form of $e^{\alpha t}$.

The displacement can now be expressed as

$$x_1 = C_1 e^{-rt/2m + (r^2/4m^2 - s/m)^{1/2} t}, \quad x_2 = C_2 e^{-rt/2m - (r^2/4m^2 - s/m)^{1/2} t}$$

or the sum of both these terms

$$x = x_1 + x_2 = C_1 e^{-rt/2m + (r^2/4m^2 - s/m)^{1/2} t} + C_2 e^{-rt/2m - (r^2/4m^2 - s/m)^{1/2} t}$$

The bracket $(r^2/4m^2 - s/m)$ can be positive, zero or negative depending on the relative magnitude of the two terms inside it. Each of these conditions gives one of the three possible solutions referred to earlier and each solution describes a particular kind of

behaviour. We shall discuss these solutions in order of *increasing* significance from our point of view; the third solution is the one we shall concentrate upon throughout the rest of this book.

The conditions are:

- (1) *Bracket positive* ($r^2/4m^2 > s/m$). Here the damping resistance term $r^2/4m^2$ dominates the stiffness term s/m , and heavy damping results in a *dead beat* system.
 (2) *Bracket zero* ($r^2/4m^2 = s/m$). The balance between the two terms results in a *critically damped* system.

Neither (1) nor (2) gives oscillatory behaviour.

- (3) *Bracket negative* ($r^2/4m^2 < s/m$). The system is lightly damped and gives oscillatory *damped simple harmonic motion*.

Case 1. Heavy Damping

Writing $r/2m = p$ and $(r^2/4m^2 - s/m)^{1/2} = q$, we can replace

$$x = C_1 e^{-rt/2m + (r^2/4m^2 - s/m)^{1/2} t} + C_2 e^{-rt/2m - (r^2/4m^2 - s/m)^{1/2} t}$$

by

$$x = e^{-pt} (C_1 e^{qt} + C_2 e^{-qt}),$$

where the C_1 and C_2 are arbitrary in value but have the same dimensions as C (note that two separate values of C are allowed because the differential equation (2.1) is second order).

If now $F = C_1 + C_2$ and $G = C_1 - C_2$, the displacement is given by

$$x = e^{-pt} \left[\frac{F}{2} (e^{qt} + e^{-qt}) + \frac{G}{2} (e^{qt} - e^{-qt}) \right]$$

or

$$x = e^{-pt} (F \cosh qt + G \sinh qt)$$

This represents non-oscillatory behaviour, but the actual displacement will depend upon the initial (or boundary) conditions; that is, the value of x at time $t = 0$. If $x = 0$ at $t = 0$ then $F = 0$, and

$$x = G e^{-rt/2m} \sinh \left(\frac{r^2}{4m^2} - \frac{s}{m} \right)^{1/2} t$$

Figure 2.2 illustrates such behaviour when a heavily damped system is disturbed from equilibrium by a sudden impulse (that is, given a velocity at $t = 0$). It will return to zero

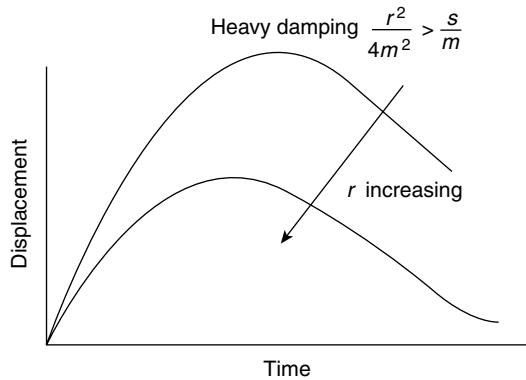


Figure 2.2 Non-oscillatory behaviour of damped simple harmonic system with heavy damping (where $r^2/4m^2 > s/m$) after the system has been given an impulse from a rest position $x = 0$

displacement quite slowly without oscillating about its equilibrium position. More advanced mathematics shows that the value of the velocity dx/dt vanishes only once so that there is only one value of maximum displacement.

(Problem 2.1)

Case 2. Critical Damping ($r^2/4m^2 = s/m$)

Using the notation of Case 1, we see that $q = 0$ and that $x = e^{-pt}(C_1 + C_2)$. This is, in fact, the limiting case of the behaviour of Case I as q changes from positive to negative. In this case the quadratic equation in α has equal roots, which, in a differential equation solution, demands that C must be written $C = A + Bt$, where A is a constant length and B a given velocity which depends on the boundary conditions. It is easily verified that the value

$$x = (A + Bt)e^{-rt/2m} = (A + Bt)e^{-pt}$$

satisfies $m\ddot{x} + r\dot{x} + sx = 0$ when $r^2/4m^2 = s/m$.

(Problem 2.2)

Application to a Damped Mechanical Oscillator

Critical damping is of practical importance in mechanical oscillators which experience sudden impulses and are required to return to zero displacement in the minimum time. Suppose such a system has zero displacement at $t = 0$ and receives an impulse which gives it an initial velocity V .

Then $x = 0$ (so that $A = 0$) and $\dot{x} = V$ at $t = 0$. However,

$$\dot{x} = B[(-pt)e^{-pt} + e^{-pt}] = B \text{ at } t = 0$$

so that $B = V$ and the complete solution is

$$x = Vt e^{-pt}$$

The maximum displacement x occurs when the system comes to rest before returning to zero displacement. At maximum displacement

$$\dot{x} = V e^{-pt}(1 - pt) = 0$$

thus giving $(1 - pt) = 0$, i.e. $t = 1/p$.

At this time the displacement is therefore

$$\begin{aligned} x &= Vt e^{-pt} = \frac{V}{p} e^{-1} \\ &= 0.368 \frac{V}{p} = 0.368 \frac{2mV}{r} \end{aligned}$$

The curve of displacement versus time is shown in Figure 2.3; the return to zero in a critically damped system is reached in *minimum* time.

Case 3. Damped Simple Harmonic Motion

When $r^2/4m^2 < s/m$ the damping is light, and this gives from the present point of view the most important kind of behaviour, *oscillatory damped simple harmonic motion*.

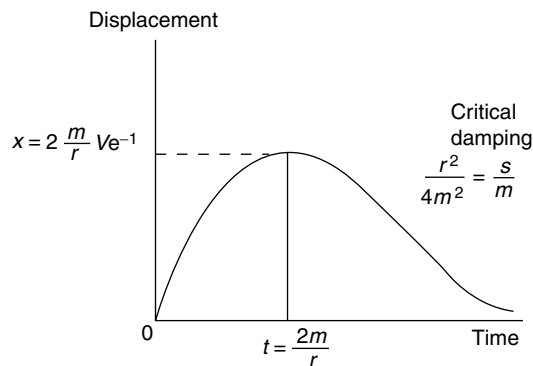


Figure 2.3 Limiting case of non-oscillatory behaviour of damped simple harmonic system where $r^2/4m^2 = s/m$ (critical damping)

The expression $(r^2/4m^2 - s/m)^{1/2}$ is an imaginary quantity, the square root of a negative number, which can be rewritten

$$\begin{aligned} \pm \left(\frac{r^2}{4m^2} - \frac{s}{m} \right)^{1/2} &= \pm \sqrt{-1} \left(\frac{s}{m} - \frac{r^2}{4m^2} \right)^{1/2} \\ &= \pm i \left(\frac{s}{m} - \frac{r^2}{4m^2} \right)^{1/2} \quad (\text{where } i = \sqrt{-1}) \end{aligned}$$

so the displacement

$$x = C_1 e^{-rt/2m} e^{+i(s/m - r^2/4m^2)^{1/2}t} + C_2 e^{-rt/2m} e^{-i(s/m - r^2/4m^2)^{1/2}t}$$

The bracket has the dimensions of inverse time; that is, of frequency, and can be written $(s/m - r^2/4m^2)^{1/2} = \omega'$, so that the second exponential becomes $e^{i\omega't} = \cos \omega't + i \sin \omega't$. This shows that the behaviour of the displacement x is oscillatory with a new frequency $\omega' < \omega = (s/m)^{1/2}$, the frequency of ideal simple harmonic motion. To compare the behaviour of the damped oscillator with the ideal case we should like to express the solution in a form similar to $x = A \sin(\omega't + \phi)$ as in the ideal case, where ω has been replaced by ω' .

We can do this by writing

$$x = e^{-rt/2m} (C_1 e^{i\omega't} + C_2 e^{-i\omega't})$$

If we now choose

$$C_1 = \frac{A}{2i} e^{i\phi}$$

and

$$C_2 = -\frac{A}{2i} e^{-i\phi}$$

where A and ϕ (and thus $e^{i\phi}$) are constants which depend on the motion at $t = 0$, we find after substitution

$$\begin{aligned} x &= A e^{-rt/2m} \frac{[e^{i(\omega't + \phi)} - e^{-i(\omega't + \phi)}]}{2i} \\ &= A e^{-rt/2m} \sin(\omega't + \phi) \end{aligned}$$

This procedure is equivalent to imposing the boundary condition $x = A \sin \phi$ at $t = 0$ upon the solution for x . The displacement therefore varies sinusoidally with time as in the case of simple harmonic motion, but now has a new frequency

$$\omega' = \left(\frac{s}{m} - \frac{r^2}{4m^2} \right)^{1/2}$$

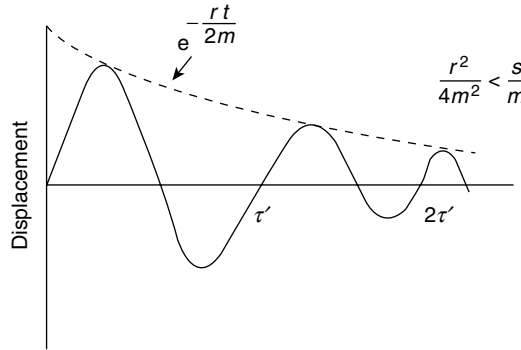


Figure 2.4 Damped oscillatory motion where $s/m > r^2/4m^2$. The amplitude decays with $e^{-rt/2m}$, and the reduced angular frequency is given by $\omega'^2 = s/m - r^2/4m^2$

and its amplitude A is modified by the exponential term $e^{-rt/2m}$, a term which decays with time.

If $x = 0$ at $t = 0$ then $\phi = 0$; Figure 2.4 shows the behaviour of x with time, its oscillations gradually decaying with the envelope of maximum amplitudes following the dotted curve $e^{-rt/2m}$. The constant A is obviously the value to which the amplitude would have risen at the first maximum if no damping were present.

The presence of the force term $r\dot{x}$ in the equation of motion therefore introduces a loss of energy which causes the amplitude of oscillation to decay with time as $e^{-rt/2m}$.

(Problem 2.3)

Methods of Describing the Damping of an Oscillator

Earlier in this chapter we saw that the energy of an oscillator is given by

$$E = \frac{1}{2}ma^2\omega^2 = \frac{1}{2}sa^2$$

that is, proportional to the square of its amplitude.

We have just seen that in the presence of a damping force $r\dot{x}$ the amplitude decays with time as

$$e^{-rt/2m}$$

so that the energy decay will be proportional to

$$(e^{-rt/2m})^2$$

that is, $e^{-rt/m}$. The larger the value of the damping force r the more rapid the decay of the amplitude and energy. Thus we can use the exponential factor to express the rates at which the amplitude and energy are reduced.

Logarithmic Decrement

This measures the rate at which the *amplitude* dies away. Suppose in the expression

$$x = A e^{-rt/2m} \sin(\omega't + \phi)$$

we choose

$$\phi = \pi/2$$

and we write

$$x = A_0 e^{-rt/2m} \cos \omega't$$

with $x = A_0$ at $t = 0$. Its behaviour will follow the curve in Figure 2.5.

If the period of oscillation is τ' where $\omega' = 2\pi/\tau'$, then one period later the amplitude is given by

$$A_1 = A_0 e^{(-r/2m)\tau'}$$

so that

$$\frac{A_0}{A_1} = e^{r\tau'/2m} = e^\delta$$

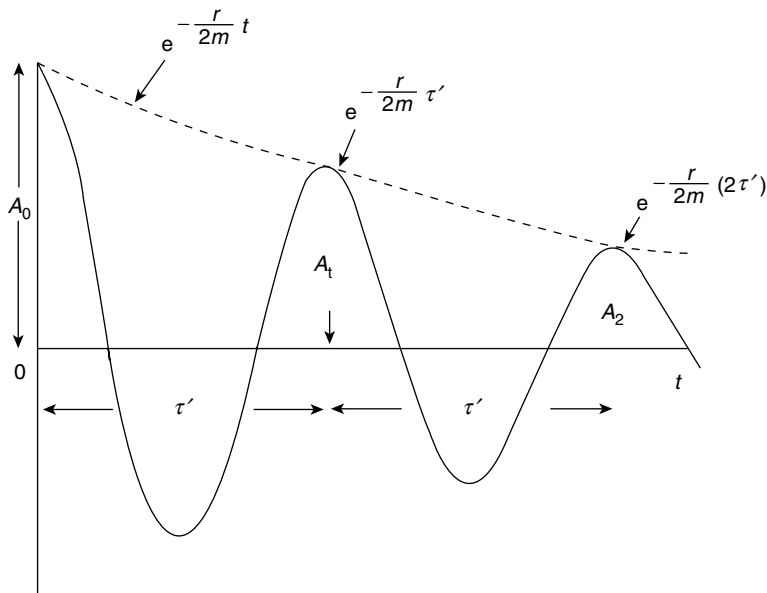


Figure 2.5 The logarithmic ratio of any two amplitudes one period apart is the logarithmic decrement, defined as $\delta = \log_e(A_n/A_{n+1}) = r\tau'/2m$

where

$$\delta = \frac{r}{2m} \tau' = \log_e \frac{A_0}{A_1}$$

is called the *logarithmic decrement*. (Note that this use of δ differs from that in Figure 1.11). The logarithmic decrement δ is the logarithm of the ratio of two amplitudes of oscillation which are separated by one period, the larger amplitude being the numerator since $e^\delta > 1$.

Similarly

$$\frac{A_0}{A_2} = e^{r(2\tau')/2m} = e^{2\delta}$$

and

$$\frac{A_0}{A_n} = e^{n\delta}$$

Experimentally, the value of δ is best found by comparing amplitudes of oscillations which are separated by n periods. The graph of

$$\log_e \frac{A_0}{A_n}$$

versus n for different values of n has a slope δ .

Relaxation Time or Modulus of Decay

Another way of expressing the damping effect is by means of the time taken for the amplitude to decay to

$$e^{-1} = 0.368$$

of its original value A_0 . This time is called the *relaxation time* or *modulus of decay* and the amplitude

$$A_t = A_0 e^{-rt/2m} = A_0 e^{-1}$$

at a time $t = 2m/r$.

Measuring the natural decay in terms of the fraction e^{-1} of the original value is a very common procedure in physics. The time for a natural decay process to reach zero is, of course, theoretically infinite.

(Problem 2.4)

The Quality Factor or Q-value of a Damped Simple Harmonic Oscillator

This measures the rate at which the *energy* decays. Since the decay of the amplitude is represented by

$$A = A_0 e^{-rt/2m}$$

the decay of energy is proportional to

$$A^2 = A_0^2 e^{(-rt/2m)^2}$$

and may be written

$$E = E_0 e^{(-r/m)t}$$

where E_0 is the energy value at $t = 0$.

The time for the energy E to decay to $E_0 e^{-1}$ is given by $t = m/r$ s during which time the oscillator will have vibrated through $\omega' m/r$ rad.

We define the *quality factor*

$$Q = \frac{\omega' m}{r}$$

as the *number of radians through which the damped system oscillates as its energy decays to*

$$E = E_0 e^{-1}$$

If r is small, then Q is very large and

$$\frac{s}{m} \gg \frac{r^2}{4m^2}$$

so that

$$\omega' \approx \omega_0 = \left(\frac{s}{m}\right)^{1/2}$$

Thus, we write, to a very close approximation,

$$Q = \frac{\omega_0 m}{r}$$

which is a constant of the damped system.

Since r/m now equals ω_0/Q we can write

$$E = E_0 e^{(-r/m)t} = E_0 e^{-\omega_0 t/Q}$$

The fact that Q is a constant ($= \omega_0 m/r$) implies that the ratio

$$\frac{\text{energy stored in system}}{\text{energy lost per cycle}}$$

is also a constant, for

$$\frac{Q}{2\pi} = \frac{\omega_0 m}{2\pi r} = \frac{\nu_0 m}{r}$$

is the number of *cycles* (or complete oscillations) through which the system moves in decaying to

$$E = E_0 e^{-1}$$

and if

$$E = E_0 e^{(-r/m)t}$$

the energy lost per cycle is

$$-\Delta E = \frac{dE}{dt} \Delta t = \frac{-r}{m} E \frac{1}{\nu'}$$

where $\Delta t = 1/\nu' = \tau'$, the period of oscillation.

Thus, the ratio

$$\begin{aligned} \frac{\text{energy stored in system}}{\text{energy lost per cycle}} &= \frac{E}{-\Delta E} = \frac{\nu' m}{r} \approx \frac{\nu_0 m}{r} \\ &= \frac{Q}{2\pi} \end{aligned}$$

In the next chapter we shall meet the same quality factor Q in two other roles, the first as a measure of the power absorption bandwidth of a damped oscillator driven near its resonant frequency and again as the factor by which the displacement of the oscillator is amplified at resonance.

Example on the Q -value of a Damped Simple Harmonic Oscillator

An electron in an atom which is freely radiating power behaves as a damped simple harmonic oscillator.

If the radiated power is given by $P = q^2 \omega^4 x_0^2 / 12\pi\epsilon_0 c^3$ W at a wavelength of $0.6 \mu\text{m}$ (6000 \AA), show that the Q -value of the atom is about 10^8 and that its free radiation lifetime is about 10^{-8} s (the time for its energy to decay to e^{-1} of its original value).

$$\begin{aligned} q &= 1.6 \times 10^{-19} \text{ C} \\ 1/4\pi\epsilon_0 &= 9 \times 10^9 \text{ m F}^{-1} \\ m_e &= 9 \times 10^{-31} \text{ kg} \\ c &= 3 \times 10^8 \text{ m s}^{-1} \\ x_0 &= \text{maximum amplitude of oscillation} \end{aligned}$$

The radiated power P is $-\nu \Delta E$, where $-\Delta E$ is the energy loss per cycle, and the energy of the oscillator is given by $E = \frac{1}{2} m_e \omega^2 x_0^2$.

Thus, $Q = 2\pi E / -\Delta E = \nu\pi m_e \omega^2 x_0^2 / P$, and inserting the values above with $\omega = 2\pi\nu = 2\pi c / \lambda$, where the wavelength λ is given, yields a Q value of $\sim 5 \times 10^7$.

The relation $Q = \omega t$ gives t , the radiation lifetime, a value of $\sim 10^{-8}$ s.

Energy Dissipation

We have seen that the presence of the resistive force reduces the amplitude of oscillation with time as energy is dissipated.

The total energy remains the sum of the kinetic and potential energies

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2$$

Now, however, dE/dt is not zero but negative because energy is lost, so that

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt} \left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2 \right) = \dot{x}(m\ddot{x} + sx) \\ &= \dot{x}(-r\dot{x}) \quad \text{for} \quad m\ddot{x} + r\dot{x} + sx = 0 \end{aligned}$$

i.e. $dE/dt = -r\dot{x}^2$, which is the rate of doing work against the frictional force (dimensions of force \times velocity = force \times distance/time).

(Problems 2.5, 2.6)

Damped SHM in an Electrical Circuit

The force equation in the mechanical oscillator is replaced by the voltage equation in the electrical circuit of inductance, resistance and capacitance (Figure 2.6).

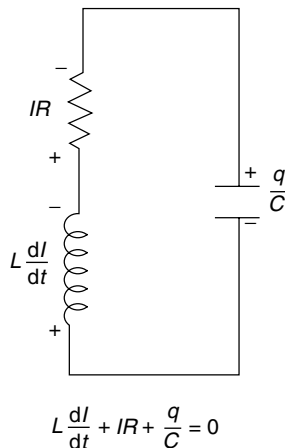


Figure 2.6 Electrical circuit of inductance, capacitance and resistance capable of damped simple harmonic oscillations. The sum of the voltages around the circuit is given from Kirchoff's law

$$\text{as } L \frac{dI}{dt} + RI + \frac{q}{C} = 0$$

We have, therefore,

$$L \frac{dI}{dt} + RI + \frac{q}{C} = 0$$

or

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0$$

and by comparison with the solutions for x in the mechanical case we know immediately that the charge

$$q = q_0 e^{-Rt/2L \pm (R^2/4L^2 - 1/LC)^{1/2} t}$$

which, for $1/LC > R^2/4L^2$, gives oscillatory behaviour at a frequency

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

From the exponential decay term we see that R/L has the dimensions of inverse time T^{-1} or ω , so that ωL has the dimensions of R ; that is, ωL is measured in ohms.

Similarly, since $\omega^2 = 1/LC$, $\omega L = 1/\omega C$, so that $1/\omega C$ is also measured in ohms. We shall use these results in the next chapter.

(Problems 2.7, 2.8, 2.9)

Problem 2.1

The heavily damped simple harmonic system of Figure 2.2 is displaced a distance F from its equilibrium position and released from rest. Show that in the expression for the displacement

$$x = e^{-pt}(F \cosh qt + G \sinh qt)$$

where

$$p = \frac{r}{2m} \quad \text{and} \quad q = \left(\frac{r^2}{4m^2} - \frac{s}{m} \right)^{1/2}$$

that the ratio

$$\frac{G}{F} = \frac{r}{(r^2 - 4ms)^{1/2}}$$

Problem 2.2

Verify that the solution

$$x = (A + Bt)e^{-rt/2m}$$

satisfies the equation

$$m\ddot{x} + r\dot{x} + sx = 0$$

when

$$r^2/4m^2 = s/m$$

Problem 2.3

The solution for damped simple harmonic motion is given by

$$x = e^{-rt/2m}(C_1 e^{i\omega't} + C_2 e^{-i\omega't})$$

If $x = A \cos \phi$ at $t = 0$, find the values of C_1 and C_2 to show that $\dot{x} \approx -\omega'A \sin \phi$ at $t = 0$ only if r/m is very small or $\phi \approx \pi/2$.

Problem 2.4

A capacitance C with a charge q_0 at $t = 0$ discharges through a resistance R . Use the voltage equation $q/C + IR = 0$ to show that the relaxation time of this process is RC s; that is,

$$q = q_0 e^{-t/RC}$$

(Note that t/RC is non-dimensional.)

Problem 2.5

The frequency of a damped simple harmonic oscillator is given by

$$\omega'^2 = \frac{s}{m} - \frac{r^2}{4m^2} = \omega_0^2 - \frac{r^2}{4m^2}$$

- If $\omega_0^2 - \omega'^2 = 10^{-6}\omega_0^2$ show that $Q = 500$ and that the logarithmic decrement $\delta = \pi/500$.
- If $\omega_0 = 10^6$ and $m = 10^{-10}$ Kg show that the stiffness of the system is 100 N m^{-1} , and that the resistive constant r is $2 \times 10^{-7} \text{ N} \cdot \text{sm}^{-1}$.
- If the maximum displacement at $t = 0$ is 10^{-2} m, show that the energy of the system is 5×10^{-3} J and the decay to e^{-1} of this value takes 0.5 ms.
- Show that the energy loss in the first cycle is $2\pi \times 10^{-5}$ J.

Problem 2.6

Show that the fractional change in the resonant frequency $\omega_0(\omega_0^2 = s/m)$ of a damped simple harmonic mechanical oscillator is $\approx (8Q^2)^{-1}$ where Q is the quality factor.

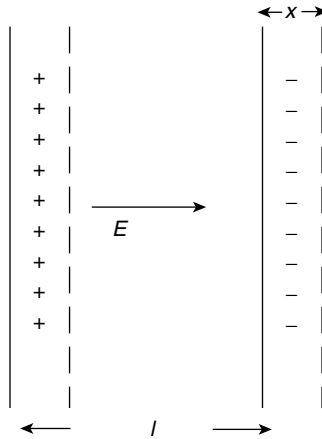
Problem 2.7

Show that the quality factor of an electrical LCR series circuit is $Q = \omega_0 L/R$ where $\omega_0^2 = 1/LC$

Problem 2.8

A plasma consists of an ionized gas of ions and electrons of equal number densities ($n_i = n_e = n$) having charges of opposite sign $\pm e$, and masses m_i and m_e , respectively, where $m_i > m_e$. Relative

displacement between the two species sets up a restoring



electric field which returns the electrons to equilibrium, the ions being considered stationary. In the diagram, a plasma slab of thickness l has all its electrons displaced a distance x to give a restoring electric field $E = nex/\epsilon_0$, where ϵ_0 is constant. Show that the restoring force per unit area on the electrons is xn^2e^2l/ϵ_0 and that they oscillate simple harmonically with angular frequency $\omega_e^2 = ne^2/m_e\epsilon_0$. This frequency is called the electron plasma frequency, and only those radio waves of frequency $\omega > \omega_e$ will propagate in such an ionized medium. Hence the reflection of such waves from the ionosphere.

Problem 2.9

A simple pendulum consists of a mass m at the end of a string of length l and performs small oscillations. The length is very slowly shortened whilst the pendulum oscillates many times at a constant amplitude $l\theta$ where θ is very small. Show that if the length is changed by $-\Delta l$ the work done is $-mg \Delta l$ (owing to the elevation of the position of equilibrium) together with an increase in the pendulum energy

$$\Delta E = \left(mg \frac{\overline{\theta^2}}{2} - ml\overline{\theta^2} \right) \Delta l$$

where $\overline{\theta^2}$ is the average value of θ^2 during the shortening. If $\theta = \theta_0 \cos \omega t$, show that the energy of the pendulum at any instant may be written

$$E = \frac{ml^2\omega^2\theta_0^2}{2} = \frac{mgl\theta_0^2}{2}$$

and hence show that

$$\frac{\Delta E}{E} = -\frac{1}{2} \frac{\Delta l}{l} = \frac{\Delta \nu}{\nu}$$

that is, E/ν , the ratio of the energy of the pendulum to its frequency of oscillation remains constant during the slowly changing process. (This constant ratio under slowly varying conditions is important in quantum theory where the constant is written as a multiple of Planck's constant, h .)

Summary of Important Results

Damped Simple Harmonic Motion

Equation of motion $m\ddot{x} + r\dot{x} + sx = 0$

Oscillations when

$$\frac{s}{m} > \frac{r^2}{4m^2}$$

Displacement $x = A e^{-rt/2m} \cos(\omega' t + \phi)$ where

$$\omega'^2 = \frac{s}{m} - \frac{r^2}{4m^2}$$

Amplitude Decay

Logarithmic decrement δ —the logarithm of the ratio of two successive amplitudes one period τ' apart

$$\delta = \log_e \frac{A_n}{A_{n+1}} = \frac{r\tau'}{2m}$$

Relaxation Time

Time for amplitude to decay to $A = A_0 e^{-rt/2m} = A_0 e^{-1}$; that is, $t = 2m/r$

Energy Decay

Quality factor Q is the number of radians during which energy decreases to $E = E_0 e^{-1}$

$$Q = \frac{\omega_0 m}{r} = 2\pi \frac{\text{energy stored in system}}{\text{energy lost per cycle}}$$

$$E = E_0 e^{-rt/m} = E_0 e^{-1} \quad \text{when } Q = \omega_0 t$$

In damped SHM

$$\frac{dE}{dt} = (m\ddot{x} + sx)\dot{x} = -r\dot{x}^2 \quad (\text{work rate of resistive force})$$

For equivalent expressions in electrical oscillators replace m by L , r by R and s by $1/C$. Force equations become voltage equations.

3

The Forced Oscillator

The Operation of i upon a Vector

We have already seen that a harmonic oscillation can be conveniently represented by the form $e^{i\omega t}$. In addition to its mathematical convenience i can also be used as a vector operator of physical significance. We say that when i precedes or operates on a vector the direction of that vector is turned through a positive angle (anticlockwise) of $\pi/2$, i.e. i acting as an operator advances the phase of a vector by 90° . The operator $-i$ rotates the vector clockwise by $\pi/2$ and retards its phase by 90° . The mathematics of i as an operator differs in no way from its use as $\sqrt{-1}$ and from now on it will play both roles.

The vector $\mathbf{r} = \mathbf{a} + i\mathbf{b}$ is shown in Figure 3.1, where the direction of \mathbf{b} is perpendicular to that of \mathbf{a} because it is preceded by i . The magnitude or modulus of \mathbf{r} is written

$$r = |\mathbf{r}| = (a^2 + b^2)^{1/2}$$

and

$$r^2 = (a^2 + b^2) = (\mathbf{a} + i\mathbf{b})(\mathbf{a} - i\mathbf{b}) = \mathbf{r}\mathbf{r}^*,$$

where $(\mathbf{a} - i\mathbf{b}) = \mathbf{r}^*$ is defined as the complex conjugate of $(\mathbf{a} + i\mathbf{b})$; that is, the sign of i is changed.

The vector $\mathbf{r}^* = \mathbf{a} - i\mathbf{b}$ is also shown in Figure 3.1.

The vector \mathbf{r} can be written as a product of its magnitude r (scalar quantity) and its phase or direction in the form (Figure 3.1)

$$\begin{aligned}\mathbf{r} &= r e^{i\phi} = r(\cos \phi + i \sin \phi) \\ &= \mathbf{a} + i\mathbf{b}\end{aligned}$$

showing that $a = r \cos \phi$ and $b = r \sin \phi$.

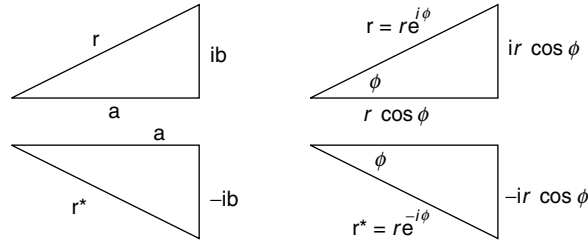


Figure 3.1 Vector representation using i operator and exponential index. Star superscript indicates complex conjugate where $-i$ replaces i

It follows that

$$\cos \phi = \frac{a}{r} = \frac{a}{(a^2 + b^2)^{1/2}}$$

and

$$\sin \phi = \frac{b}{r} = \frac{b}{(a^2 + b^2)^{1/2}}$$

giving $\tan \phi = b/a$.

Similarly

$$\mathbf{r}^* = r e^{-i\phi} = r(\cos \phi - i \sin \phi)$$

$$\cos \phi = \frac{a}{r}, \quad \sin \phi = \frac{-b}{r} \quad \text{and} \quad \tan \phi = \frac{-b}{a} \quad (\text{Figure 3.1})$$

The reader should confirm that the operator i rotates a vector by $\pi/2$ in the positive direction (as stated in the first paragraph of p. 53) by taking $\phi = \pi/2$ in the expression

$$\mathbf{r} = r e^{i\phi} = r(\cos \pi/2 + i \sin \pi/2)$$

Note that $\phi = -\pi/2$ in $\mathbf{r} = r e^{-i\pi/2}$ rotates the vector in the negative direction.

Vector form of Ohm's Law

Ohm's Law is first met as the scalar relation $V = IR$, where V is the voltage across the resistance R and I is the current through it. Its scalar form states that the voltage and current are always in phase. Both will follow a $\sin(\omega t + \phi)$ or a $\cos(\omega t + \phi)$ curve, and the value of ϕ will be the same for both voltage and current.

However, the presence of either or both of the other two electrical components, inductance L and capacitance C , will introduce a phase difference between voltage and

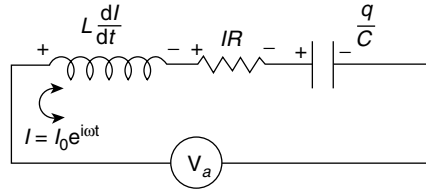


Figure 3.2a An electrical forced oscillator. The voltage V_a is applied to the series LCR circuit giving $V_a = LdI/dt + IR + q/C$

current, and Ohm's Law takes the vector form

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_e,$$

where \mathbf{Z}_e , called the *impedance*, replaces the resistance, and is the vector sum of the effective resistances of R , L , and C in the circuit.

When an alternating voltage V_a of frequency ω is applied across a resistance, inductance and condenser in series as in Figure 3.2a, the balance of voltages is given by

$$V_a = IR + L \frac{dI}{dt} + q/C$$

and the current through the circuit is given by $I = I_0 e^{i\omega t}$. The voltage across the inductance

$$V_L = L \frac{dI}{dt} = L \frac{d}{dt} I_0 e^{i\omega t} = i\omega L I_0 e^{i\omega t} = i\omega L I$$

But ωL , as we saw at the end of the last chapter, has the dimensions of ohms, being the value of the effective resistance presented by an inductance L to a current of frequency ω . The product $\omega L I$ with dimensions of ohms times current, i.e. volts, is preceded by i ; this tells us that the phase of the voltage across the inductance is 90° ahead of that of the current through the circuit.

Similarly, the voltage across the condenser is

$$\frac{q}{C} = \frac{1}{C} \int I dt = \frac{1}{C} I_0 \int e^{i\omega t} dt = \frac{1}{i\omega C} I_0 e^{i\omega t} = -\frac{iI}{\omega C}$$

(since $1/i = -i$).

Again $1/\omega C$, measured in ohms, is the value of the effective resistance presented by the condenser to the current of frequency ω . Now, however, the voltage $I/\omega C$ across the condenser is preceded by $-i$ and therefore lags the current by 90° . The voltage and current across the resistance are in phase and Figure 3.2b shows that the vector form of Ohm's Law may be written $\mathbf{V} = \mathbf{I}\mathbf{Z}_e = I[R + i(\omega L - 1/\omega C)]$, where the impedance $\mathbf{Z}_e = R + i(\omega L - 1/\omega C)$. The quantities ωL and $1/\omega C$ are called *reactances* because they

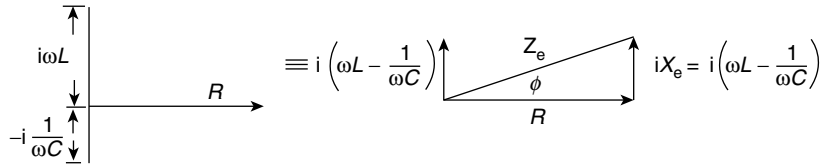


Figure 3.2b Vector addition of resistance and reactances to give the electrical impedance $\mathbf{Z}_e = R + i(\omega L - 1/\omega C)$

introduce a phase relationship as well as an effective resistance, and the bracket $(\omega L - 1/\omega C)$ is often written X_e , the reactive component of \mathbf{Z}_e .

The magnitude, in ohms, i.e. the value of the impedance, is

$$Z_e = \left[R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2 \right]^{1/2}$$

and the vector \mathbf{Z}_e may be represented by its magnitude and phase as

$$\mathbf{Z}_e = Z_e e^{i\phi} = Z_e (\cos \phi + i \sin \phi)$$

so that

$$\cos \phi = \frac{R}{Z_e}, \quad \sin \phi = \frac{X_e}{Z_e}$$

and

$$\tan \phi = X_e/R,$$

where ϕ is the phase difference between the total voltage across the circuit and the current through it.

The value of ϕ can be positive or negative depending on the relative value of ωL and $1/\omega C$: when $\omega L > 1/\omega C$, ϕ is positive, but the frequency dependence of the components show that ϕ can change both sign and size.

The magnitude of Z_e is also frequency dependent and has its minimum value $Z_e = R$ when $\omega L = 1/\omega C$.

In the vector form of Ohm's Law, $\mathbf{V} = \mathbf{I}\mathbf{Z}_e$. If $\mathbf{V} = V_0 e^{i\omega t}$ and $\mathbf{Z}_e = Z_e e^{i\phi}$, then we have

$$\mathbf{I} = \frac{V_0 e^{i\omega t}}{Z_e e^{i\phi}} = \frac{V_0}{Z_e} e^{i(\omega t - \phi)}$$

giving a current of amplitude V_0/Z_e which lags the voltage by a phase angle ϕ .

The Impedance of a Mechanical Circuit

Exactly similar arguments hold when we consider not an electrical oscillator but a mechanical circuit having mass, stiffness and resistance.